

MULTIVARIATE ANALYSIS OF NONPARAMETRIC ESTIMATES OF LARGE CORRELATION MATRICES

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We study concentration in spectral norm of nonparametric estimates of correlation matrices. We work within the confine of a Gaussian copula model. Two nonparametric estimators of the correlation matrix, the sine transformations of the Kendall's tau and Spearman's rho correlation coefficient, are studied. Expected spectrum error bound is obtained for both the estimators. A general large deviation bound for the maximum spectral error of a collection of submatrices of a given dimension is also established. These results prove that when both the number of variables and sample size are large, the spectral error of the nonparametric estimators is of no greater order than that of the latent sample covariance matrix, at least when compared with some of the sharpest known error bounds for the later. As an application, we establish the minimax optimal convergence rate in the estimation of high-dimensional bandable correlation matrices via tapering off of these nonparametric estimators. An optimal convergence rate for sparse principal component analysis is also established as another example of possible applications of the main results.

1. Introduction. We consider n iid copies $\{\mathbf{X}_i : 1 \leq i \leq n\}$ of a d -dimensional Gaussian random vector $(X_1, \dots, X_d)^T$. We define $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_n)^T \in \mathbb{R}^{n \times d}$. We assume that \mathbf{X}_i 's are centered and marginally scaled, so that $\mathbb{E}\mathbf{X} = \mathbf{0}$ and the correlation matrix is given by $\mathbb{E}\mathbf{X}\mathbf{X}^T/n = \Sigma \in \mathbb{R}^{d \times d}$ with 1 in the diagonal. In this paper, we work within a high-dimensional 'double asymptotic' setting where $d \wedge n \rightarrow \infty$. We assume that instead of \mathbf{X} , we only observe n iid copies $\mathbf{Y}_i, 1 \leq i \leq n$, of the transformed variables

$$(f_1(X_1), \dots, f_d(X_d))^T$$

where f_i 's are unknown but strictly increasing. This is a form of the copula model (Sklar, 1959) for the distribution of the data. Because \mathbf{X} follows a Gaussian distribution, it is a formulation of the Gaussian copula, cf. Bickel et al. (1993) and references therein. A slightly different but equivalent formulation of the Gaussian copula has been referred to as the *nonparanormal* model (Liu, Lafferty and Wasserman, 2009). Let $\mathbf{Y} = (\mathbf{Y}_1, \dots, \mathbf{Y}_n)^T$. Our goal here is to estimate the latent correlation structure Σ using the observed data matrix \mathbf{Y} .

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If we could observe the latent data matrix \mathbf{X} , an obvious choice as an estimator would be the sample correlation matrix given by $\tilde{\Sigma}^s = \mathbf{X}^T \mathbf{X} / n$. It is for this reason that we refer to the latent $\tilde{\Sigma}^s$ as an oracle estimator. It is also clear that $\tilde{\Sigma}^s$ is a sufficient statistic for estimating Σ when \mathbf{X} is known. As a consequence, any statistical procedure based on Σ could be summarily described as $g(\tilde{\Sigma}^s)$ for some function g . In this respect, $\tilde{\Sigma}^s$ possesses great utility as an ideal raw estimate that lends itself to further analysis as the need be.

However, as noted above, we do not observe \mathbf{X} but unknown strictly monotone transformations of columns of it, \mathbf{Y} . Thus the sample correlation matrix based on \mathbf{Y} , i.e. $\mathbf{Y}^T \mathbf{Y} / n$, is in general inconsistent in estimating the latent correlation structure Σ . Two candidate nonparametric estimators in such a scenario are considered in this paper: Kendall's tau developed in [Kendall \(1938\)](#) and Spearman's rank correlation coefficient, developed by Charles Spearman in 1904. These are two widely used nonparametric measures of association. Their properties in fixed dimension have been studied in [Kendall \(1938, 1948\)](#), [Kruskal \(1958\)](#) and many others. More recently, in high-dimensional scenarios, correlation matrix estimators based on these measures have been taken up for study in [Liu et al. \(2012\)](#) and [Xue and Zou \(2012a\)](#) among others.

For the rest of this paper, we call $\hat{\Sigma}^\tau$ the correlation matrix estimator based on Kendall's tau and call $\hat{\Sigma}^\rho$ the one based on Spearman's rho. It will be interesting to study whether for any statistical procedure, say $g(\tilde{\Sigma}^s)$, based on the raw estimate $\tilde{\Sigma}^s$, it is possible to provide justification for the use of $g(\hat{\Sigma}^\tau)$ or $g(\hat{\Sigma}^\rho)$ as a viable replacement. It is however cumbersome to study each individual procedure separately. On the other hand, if g is sufficiently smooth with respect to some matrix norm, it would suffice to study the accuracy of $\hat{\Sigma}^\tau$ and $\hat{\Sigma}^\rho$ as estimates of Σ in such norms.

A complete description of properties of $\hat{\Sigma}^\tau$ and $\hat{\Sigma}^\rho$ as estimators of large Σ necessitates the derivation of the distributions of these matrix estimators. It is well known that in the multivariate Gaussian model, $\tilde{\Sigma}^s$ follows a Wishart distribution ([Anderson, 1958](#)). To the contrary, derivation of the distribution of $\hat{\Sigma}^\tau$ and $\hat{\Sigma}^\rho$ seems at the present moment intractable. On the other hand, analysis of these nonparametric estimators for each individual element of the correlation matrix has been taken upon before. Both Kendall's tau and Spearman's rho are specific instances of U-statistics with bounded kernels. In [Hoeffding \(1948\)](#), the asymptotic normality of these nonparametric estimators for an individual correlation was established. Furthermore, the celebrated [Hoeffding \(1963\)](#) inequality provides large deviation bounds for these estimators as U-statistics with bounded kernels. These results provide tools for studying the concentration of $\hat{\Sigma}^\tau$ and $\hat{\Sigma}^\rho$ in the matrix max norm and its applications ([Liu et al., 2012](#); [Xue and Zou, 2012a](#)) and the corresponding Gaussian copula graphical model ([Liu, Han and Zhang, 2012](#)).

It is important to note that while estimation accuracy in one specific matrix norm could be more appropriate for a certain set of statistical problems, some other set of problems might require accuracy in a different matrix norm. In this paper we focus on the spectral norm, which is also understood as the ℓ_2 operator norm. Many statistical problems can be studied with error bounds in the spectral norm of estimated correlation

matrices. A primary example is the principal component analysis (PCA) since the spectral norm is essential in studying the effects of matrix perturbation on eigenvalues and eigenvectors.

Before beginning the study of convergence of $\widehat{\Sigma}^\tau$ and $\widehat{\Sigma}^\rho$ in the spectral norm, it is worthwhile to note that convergence rate of the latent sample covariance matrix $\widetilde{\Sigma}^s$ in the spectral norm has been studied widely and established in a multitude of literature. A detailed overview and further references can be found in [Vershynin \(2010\)](#) among others. For example, one could derive, from the concentration inequality in Theorem II.13 of [Davidson and Szarek \(2001\)](#), that for $\mathbf{X} \in \mathbb{R}^{n \times d}$ with iid $N(\mathbf{0}, \Sigma)$ rows,

$$(1.1) \quad \sqrt{\mathbb{E} \|\widetilde{\Sigma}^s - \Sigma\|_S^2} \leq \|\Sigma\|_S \left(2\sqrt{2}\sqrt{d/n} + \sqrt{2}d/n + 6(d/n^3)^{1/4} \right),$$

so that the consistency of $\widetilde{\Sigma}^s$ follows when $d/n \rightarrow 0$. Additionally, the concentration inequality also provides a uniform bound on the spectral error for any s -dimensional diagonal submatrix for larger d . Taking any integer $s < d$ and sets $A \subset \{1, \dots, d\}$, we have by the union bound

$$(1.2) \quad \begin{aligned} & \max_{|A| \leq s} \|(\widetilde{\Sigma}^s - \Sigma)_{A \times A}\|_S / \max_{|A| \leq s} \|\Sigma_{A \times A}\|_S \\ & \leq \left(\sqrt{s/n} + \sqrt{2\{t + \log \binom{d}{s}\}/n} \right) \left(2 + \sqrt{s/n} + \sqrt{2\{t + \log \binom{d}{s}\}/n} \right) \end{aligned}$$

with at least probability $1 - 2e^{-t}$. These spectral error bounds are explicit and of sharp order for the latent sample correlation matrix estimate $\widetilde{\Sigma}^s$. In this light, it is apt to ask whether $\widehat{\Sigma}^\tau$ and $\widehat{\Sigma}^\rho$ also submit similar error bounds.

In [Han and Liu \(2013\)](#) a rate of $\sqrt{d \log d/n}$ was established for $\widehat{\Sigma}^\tau$ in a transelliptical family of distributions ([Liu, Han and Zhang, 2012](#)). In a separate but simultaneous work in [Wegkamp and Zhao \(2013\)](#) the same rate was established for $\widehat{\Sigma}^\tau$ in an elliptical copula correlation factor model, which can be also viewed as elliptical copula. In this paper, we provide non-asymptotic spectrum error bounds in the more restrictive Gaussian copula model for both $\widehat{\Sigma}^\tau$ and $\widehat{\Sigma}^\rho$ which improve the convergence rates of these existing error bounds. In particular, we establish in Theorem 1 expected spectral error bounds to match (1.1), and under mild conditions on the sample size, we establish in Theorem 2 and its corollaries large deviation bounds to match (1.2). These results establish that in the Gaussian copula model the nonparametric estimators $\widehat{\Sigma}^\tau$ and $\widehat{\Sigma}^\rho$ perform as well as the oracle raw estimator $\widetilde{\Sigma}^s$ in terms of the order of the spectral error. Consequently, a methodology based on $\widetilde{\Sigma}^s$ that hinges on a spectrum error bound can be performed with the same rate of convergence if $\widehat{\Sigma}^\tau$ or $\widehat{\Sigma}^\rho$ are used in lieu of the latent $\widetilde{\Sigma}^s$.

We discuss two different statistical problems where our results could be applied. The first, a ripe problem for application of spectral error bounds, is the estimation of a large bandable correlation matrix. For high-dimensional data, proper estimation of large bandable Σ involves implementation of various regularization strategies such as banding, tapering, thresholding etc. These procedures and their properties have been

studied in Wu and Pourahmadi (2003), Bickel and Levina (2008a,b), Karoui (2008), Lam and Fan (2009), Cai and Liu (2011), Cai and Zhou (2012), and Cai and Yuan (2012). In particular, Cai, Zhang and Zhou (2010) established the optimal minimax rate of convergence for a tapered version of $\tilde{\Sigma}^s$ for certain classes of unknown bandable Σ . In Xue and Zou (2012b), a tapering estimator based on the Spearman's rank correlation was studied for the same class of parameters in the Gaussian copula model. However, the question of whether the nonparametric estimator could attain the optimal rate, was not resolved in their paper. Our spectral error bounds imply that the optimal rate is attained if one substitutes $\tilde{\Sigma}^s$ with either $\hat{\Sigma}^\tau$ or $\hat{\Sigma}^\rho$.

The second application involves error bounds in the estimation of the leading eigenvector in PCA both with and without a sparsity assumption on the eigenvector. With the advent and increasing prevalence of high dimensional data, various limitations of traditional procedures had come to the fore. For instance, Johnstone and Lu (2009) showed that when $d/n \rightarrow c > 0$, the principal component of $\tilde{\Sigma}^s$ is inconsistent in estimating the leading eigenvector of the true correlation matrix. Several remedies to this problem have been proposed, all being different formulations under the auspice of a general sparse PCA paradigm. In sparse PCA, the eigenvectors corresponding to the largest eigenvalues are assumed to be sparse. A vast array of sparse PCA approaches has been proposed and studied in Jolliffe, Trendafilov and Uddin (2003), Zou, Hastie and Tibshirani (2006), d'Aspremont et al. (2007), Vu and Lei (2012), Ma (2013), and Cai, Ma and Wu (2013) among others. For the elliptical copula family, Han and Liu (2013) established the optimal rate of convergence in sparse PCA with $\hat{\Sigma}^\tau$ under an additional sign sub-Gaussian condition. We will demonstrate that our spectral error bounds for the nonparametric estimators can be directly applied to study the convergence rates for the principle component direction. In particular, for sparse PCA the minimax rate as described in Vu and Lei (2012) will be established without imposing the sign sub-Gaussian condition.

Our work is organized as follows. In Section 2 we describe the Gaussian copula model and the Kendall's tau and Spearman's rho estimators for the correlation matrix. In Section 3, we provide upper bounds for the expected spectral error for these two correlation-matrix estimators in Theorem 1 and outline our analytical strategy. In Section 4, we provides a general large deviation inequality in Theorem 2. In Section 5 we discuss two problems where our results on spectral norm concentration could be utilized. Some of the proofs are relegated to the Appendix.

2. Background & Preliminary Results. We describe the basic data model and define the nonparametric estimates of Σ .

2.1. Data Model and Notation. We consider the Gaussian copula or multivariate nonparametric transformational model

$$(2.1) \quad (Y_1, \dots, Y_d)^T = (f_1(X_1), \dots, f_d(X_d))^T,$$

where $(X_1, \dots, X_d)^T \in \mathbb{R}^d$ is a multivariate Gaussian random vector with marginal $N(0, 1)$ distribution and f_j are unknown strictly increasing functions. We are interested

in estimating the population correlation matrix of $(X_1, \dots, X_d)^T$, denoted by

$$(2.2) \quad \Sigma = \mathbb{E}(X_1, \dots, X_d)^T (X_1, \dots, X_d),$$

based on a sample of iid copies of $(Y_1, \dots, Y_d)^T$. Since the f_j absorbs the location and scale of the individual X_j , it is natural to assume $\mathbb{E}X_j = 0$ and $\mathbb{E}X_j^2 = 1$ on the marginal distribution.

The observations $\mathbf{Y}_i = (Y_{i1}, \dots, Y_{id})^T$, $i = 1, \dots, n$, are iid copies of $(Y_1, \dots, Y_d)^T$. They can be written as

$$(2.3) \quad Y_{ij} = f_j(X_{ij}) \quad i = 1, \dots, n \quad j = 1, \dots, d,$$

where $\mathbf{X}_i = (X_{i1}, \dots, X_{id})^T \in \mathbb{R}^d$ are independent copies of $(X_1, \dots, X_d)^T \sim N(\mathbf{0}, \Sigma)$ in (2.1). We denote by $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_n)^T \in \mathbb{R}^{n \times d}$ the matrix with rows \mathbf{X}_i^T and quite similarly $\mathbf{Y} = (\mathbf{Y}_1, \dots, \mathbf{Y}_n)^T \in \mathbb{R}^{n \times d}$.

We use the following notation throughout the paper. For vectors $\mathbf{u} \in \mathbb{R}^d$, the ℓ_p norm is denoted by $\|\mathbf{u}\|_p = \left(\sum_{k=1}^d |u_k|^p \right)^{1/p}$, with $\|\mathbf{u}\|_\infty = \max_{1 \leq k \leq d} |u_k|$ and $\|\mathbf{u}\|_0 = \#\{j : u_j \neq 0\}$. For matrices $\mathbf{A} = (A_{jk})_{d \times d} \in \mathbb{R}^{d \times d}$, the $\ell_p \rightarrow \ell_q$ operator norm is denoted by $\|\mathbf{A}\|_{(p,q)} = \max_{\|\mathbf{u}\|_p=1} \|\mathbf{A}\mathbf{u}\|_q$. The $\ell_2 \rightarrow \ell_2$ operator norm, known as the spectrum norm, is

$$\|\mathbf{A}\|_S = \|\mathbf{A}\|_{(2,2)} = \max_{\|\mathbf{u}\|_2=1} |\mathbf{u}^T \mathbf{A} \mathbf{u}|$$

The vectorized ℓ_∞ and Frobenius norms are denoted by

$$\|\mathbf{A}\|_{\max} = \max_{j,k} |A_{jk}|, \quad \|\mathbf{A}\|_F = \sqrt{\text{trace}(\mathbf{A}^T \mathbf{A})}.$$

For symmetric matrices \mathbf{A} , the j^{th} eigenpair of \mathbf{A} is denoted by $\lambda_j(\mathbf{A})$ and $\boldsymbol{\theta}_j(\mathbf{A})$, so that $\lambda_1(\mathbf{A}) = \|\mathbf{A}\|_S$ and $\boldsymbol{\theta}_1(\mathbf{A})$ is the leading eigenvector. In addition to \mathbb{E} and \mathbb{P} , which denote the expectation and probability measure, we denote by \mathbb{E}_n the average over iid copies of variables in (2.3). For example,

$$\mathbb{E}_n h(x_j, x_k) = n^{-1} \sum_{i=1}^n h(X_{ij}, X_{ik}).$$

The relation $a_n = \mathcal{O}(b_n)$ will imply $a_n \leq K b_n$ for some fixed constant $K > 0$. Finally we denote $\mathbb{S}^{d-1} = \{\mathbf{u} \in \mathbb{R}^d : \|\mathbf{u}\|_2 = 1\}$.

2.2. Nonparametric Estimation of Correlation Matrix. The approach we adopt in estimating the correlation matrix $\Sigma = (\Sigma_{jk})$ in (2.2) is based on Kendall's tau (τ) or Spearman's correlation coefficient rho (ρ).

With the observations Y_{ij} in (2.3), Kendall's tau is defined as

$$(2.4) \quad \hat{\tau}_{jk} = \frac{2}{n(n-1)} \sum_{1 \leq i_1 < i_2 \leq n} \text{sgn}(Y_{i_1 j} - Y_{i_2 j}) \text{sgn}(Y_{i_1 k} - Y_{i_2 k}),$$

and Spearman's rho as

$$(2.5) \quad \hat{\rho}_{jk} = \frac{\sum_{i=1}^n (r_{ij} - (n+1)/2)(r_{ik} - (n+1)/2)}{\sqrt{\sum_{i=1}^n (r_{ij} - (n+1)/2)^2 \sum_{i=1}^n (r_{ik} - (n+1)/2)^2}},$$

where r_{ij} is the rank of Y_{ij} among Y_{1j}, \dots, Y_{nj} . In matrix notation,

$$(2.6) \quad \hat{\mathbf{T}} = (\hat{\tau}_{jk})_{d \times d}, \quad \hat{\mathbf{R}} = (\hat{\rho}_{jk})_{d \times d}.$$

The population version of Kendall's tau is given by

$$(2.7) \quad \tau_{jk} = \mathbb{E} \operatorname{sgn}(Y_{1j} - Y_{2j}) \operatorname{sgn}(Y_{1k} - Y_{2k}),$$

while the population version of Spearman's rho is given by

$$(2.8) \quad \rho_{jk} = 3 \mathbb{E} \operatorname{sgn}(Y_{1j} - Y_{2j}) \operatorname{sgn}(Y_{1k} - Y_{3k}).$$

In matrix notation, the population version of (2.6) is

$$(2.9) \quad \mathbf{T} = (\tau_{jk})_{d \times d}, \quad \mathbf{R} = (\rho_{jk})_{d \times d}.$$

Since f_j are strictly increasing functions, we have $\operatorname{sgn}(f_j(u) - f_j(v)) = \operatorname{sgn}(u - v)$. Thus, Kendall's tau, Spearman's rho and their population version are unchanged if the observed $\mathbf{Y} = (Y_{ij})_{n \times d}$ is replaced by the unobserved $\mathbf{X} = (X_{ij})_{n \times d}$ in their definition. Since X_j follows a standard normal distribution, we have, from [Kendall \(1948\)](#) and [Kruskal \(1958\)](#), that for $\Sigma_{jk} = \mathbb{E} X_j X_k$,

$$(2.10) \quad \Sigma_{jk} = \sin\left(\frac{\pi}{2} \tau_{jk}\right) = 2 \sin\left(\frac{\pi}{6} \rho_{jk}\right).$$

This immediately leads to the following correlation matrix estimator by Kendall's tau,

$$(2.11) \quad \hat{\Sigma}^\tau = (\hat{\Sigma}_{jk}^\tau)_{d \times d}, \quad \hat{\Sigma}_{jk}^\tau = \sin\left(\frac{\pi}{2} \hat{\tau}_{jk}\right).$$

In the same light we define the correlation matrix estimator by Spearman's rho as

$$(2.12) \quad \hat{\Sigma}^\rho = (\hat{\Sigma}_{jk}^\rho)_{d \times d}, \quad \hat{\Sigma}_{jk}^\rho = 2 \sin\left(\frac{\pi}{6} \hat{\rho}_{jk}\right).$$

The following proposition states a slightly different version of Theorem 2.3 of [Wegkamp and Zhao \(2013\)](#) and a direct application of their argument to Spearman's rho.

PROPOSITION 1. *Both matrices $\mathbf{T} - (2/\pi)\Sigma$ and $\mathbf{R} - (3/\pi)\Sigma$ are nonnegative-definite, $\|\mathbf{T} - (2/\pi)\Sigma\|_S \leq (1 - 2/\pi)\|\Sigma\|_S$, and $\|\mathbf{R} - (3/\pi)\Sigma\|_S \leq (1 - 3/\pi)\|\Sigma\|_S$. Consequently,*

$$(2.13) \quad \|\mathbf{T}\|_S \vee \|\mathbf{R}\|_S \leq \|\Sigma\|_S.$$

3. Expected Spectrum Error Bounds. While Spearman's rho and Kendall's tau are structurally different, they can be represented neatly as U-statistics of a special type. In this section we develop bounds for the expected spectrum norm of their error via a certain decomposition of such U-statistics. This decomposition also provides an outline of our analysis of the concentration of the spectrum norm and the sparse spectrum norm of the error in subsequent sections.

Given a sequence of n observations from a population in \mathbb{R}^d , a matrix U-statistic with order m and kernels $h_{jk}(\mathbf{x}_1, \dots, \mathbf{x}_m)$ can be written as

$$(3.1) \quad \mathbf{U}_n = (U_{n;jk})_{d \times d}$$

with elements

$$(3.2) \quad U_{n;jk} = \frac{(n-m)!}{n!} \sum_{1 \leq i_1 \neq \dots \neq i_m \leq n} h_{jk}(\mathbf{X}_{i_1}, \mathbf{X}_{i_2}, \dots, \mathbf{X}_{i_m}).$$

Assume that $h_{jk}(\mathbf{x}_1, \dots, \mathbf{x}_m)$ are permutation symmetric and set

$$(3.3) \quad \bar{h}_{jk}(\mathbf{x}) = \mathbb{E} \left[h_{jk}(\mathbf{X}_1, \dots, \mathbf{X}_m) \middle| \mathbf{X}_1 = \mathbf{x} \right] - c_{jk}$$

with any constants c_{jk} . The Hoeffding decomposition of \mathbf{U}_n can be written as

$$(3.4) \quad \mathbf{U}_n - \mathbb{E}\mathbf{U}_n = \sum_{\ell=1}^m \binom{m}{\ell} \Delta_n^{(\ell)}$$

where $\Delta_n^{(1)}$ is an average of iid random matrices with elements

$$(3.5) \quad \Delta_{n;jk}^{(1)} = (\mathbb{E}_n - \mathbb{E})\bar{h}_{jk} = \frac{1}{n} \sum_{i=1}^n \left(\bar{h}_{jk}(\mathbf{X}_i) - \mathbb{E}\bar{h}_{jk}(\mathbf{X}_1) \right)$$

and $\Delta_n^{(\ell)} = (\Delta_{n;jk}^{(\ell)})_{d \times d}$ are matrix U-statistics with completely degenerate kernels of order ℓ . We refer to [Hoeffding \(1948\)](#), [Hájek, Šidák and Sen \(1967\)](#), [Hájek \(1968\)](#), [Van der Vaart \(2000\)](#) and [Serfling \(2009\)](#) for detailed exposition on the Hoeffding decomposition and additional references.

Since the components of the Hoeffding decomposition are orthogonal,

$$\begin{aligned} \mathbb{E} \left(\sum_{\ell=2}^m \binom{m}{\ell} \Delta_{n;jk}^{(\ell)} \right)^2 &= \sum_{\ell=2}^m \binom{n}{\ell}^{-1} \binom{m}{\ell}^3 \mathbb{E} \left(\Delta_{m;jk}^{(\ell)} \right)^2 \\ &\leq \binom{n}{2}^{-1} \binom{m}{2} \text{Var}(h_{jk}(\mathbf{X}_1, \dots, \mathbf{X}_m)). \end{aligned}$$

A consequence of the above calculation of variance is

$$\mathbb{E} \left\| \mathbf{U}_n - \mathbb{E}\mathbf{U}_n - m\Delta_n^{(1)} \right\|_F^2 \leq \frac{m(m-1)}{n(n-1)} \sum_{j=1}^d \sum_{k=1}^d \text{Var}(h_{jk}(\mathbf{X}_1, \dots, \mathbf{X}_m)).$$

We note that Kendall's tau and Spearman's rho are U-statistics of order $m = 2$ and 3 respectively, both with kernels satisfying $h_{jj}(\mathbf{x}_1, \dots, \mathbf{x}_m) = 1$ and $\|h_{jk}(\mathbf{x}_1, \dots, \mathbf{x}_m)\|_\infty \leq 1$ for $j \neq k$. It follows that the high order terms of their Hoeffding decompositions are explicitly bounded by

$$(3.6) \quad \mathbb{E} \left\| \mathbf{U}_n - \mathbb{E} \mathbf{U}_n - m \boldsymbol{\Delta}_n^{(1)} \right\|_F^2 \leq \frac{m(m-1)d(d-1)}{n(n-1)}.$$

Now we consider the term $\boldsymbol{\Delta}_n^{(1)}$. It turns out that in the Gaussian copula model (2.3), the first order kernel for Kendall's tau can be written as

$$\bar{h}_{jk}(x_1, \dots, x_d) = \begin{cases} \bar{h}(x_j, x_k, \Sigma_{jk}), & j \neq k \\ 1 & j = k \end{cases}$$

with $\bar{h}(x_j, x_k, 0) = \bar{h}_0(x_j)\bar{h}_0(x_k)$, where $\bar{h}_0(x) = 2\Phi(x) - 1$, and that of Spearman's rho is of the same form. This motivates a further decomposition of $\boldsymbol{\Delta}_n^{(1)}$ as a sum of $\boldsymbol{\Delta}_n^{(0)}$ and $\boldsymbol{\Delta}_n^{(1)} - \boldsymbol{\Delta}_n^{(0)}$, with

$$(3.7) \quad \begin{aligned} \boldsymbol{\Delta}_n^{(0)} &= \left(\Delta_{n;jk}^{(0)} \right)_{d \times d} = \left((\mathbb{E}_n - \mathbb{E}) \bar{h}_0(x_j) \bar{h}_0(x_k) \right)_{d \times d}, \\ \boldsymbol{\Delta}_n^{(1)} - \boldsymbol{\Delta}_n^{(0)} &= \left((\mathbb{E}_n - \mathbb{E}) (\bar{h}(x_j, x_k, \Sigma_{jk}) - \bar{h}(x_j, x_k, 0)) \right)_{d \times d} \end{aligned}$$

It follows from the definition of the population Spearman's rho in (2.8) that

$$\mathbb{E} \bar{h}(X_j, X_k, 0) = \mathbb{E} \bar{h}_0(X_j) \bar{h}_0(X_k) = \rho_{jk}/3, \quad \forall 1 \leq j \leq k \leq d.$$

Thus, the $\boldsymbol{\Delta}_n^{(0)}$ in (3.7) can be written as the difference between the sample covariance matrix of $\bar{h}_0(\mathbf{X}) = (\bar{h}_0(X_{ij}))_{n \times d}$ and its expectation:

$$(3.8) \quad \boldsymbol{\Delta}_n^{(0)} = n^{-1} \bar{h}_0(\mathbf{X})^T \bar{h}_0(\mathbf{X}) - \mathbf{R}/3.$$

Moreover, we will prove that for both Kendall's tau and Spearman's rho

$$(3.9) \quad \left| \bar{h}(x_j, x_k, \Sigma_{jk}) - \bar{h}(x_j, x_k, 0) \right| \leq C_1 \left| \Sigma_{jk} \right|, \quad j \neq k.$$

with $C_1 = 2/\pi + 1 \leq 2$ for Kendall's tau and $C_1 \leq 1 + \sqrt{8}/\pi \leq 2$ for Spearman's rho. Thus, since $\text{Var}(\bar{h}_0^2(X_{ij})) = \int_0^1 ((2x-1)^2 - 1/3)^2 dx = 4/45$ on the diagonal of $\boldsymbol{\Delta}_n^{(1)} - \boldsymbol{\Delta}_n^{(0)}$ and $\boldsymbol{\Delta}_n^{(1)} - \boldsymbol{\Delta}_n^{(0)}$ is an average of iid matrices,

$$(3.10) \quad \mathbb{E} \left\| \boldsymbol{\Delta}_n^{(1)} - \boldsymbol{\Delta}_n^{(0)} \right\|_S^2 \leq \mathbb{E} \left\| \boldsymbol{\Delta}_n^{(1)} - \boldsymbol{\Delta}_n^{(0)} \right\|_F^2 \leq C_1^2 \sum_{j \neq k} \frac{\Sigma_{jk}^2}{n} + \frac{4d}{45n}.$$

Let \mathbf{U}_n be the matrix U-statistics of either Kendall's tau or Spearman's rho, $\mathbf{U}_n = \widehat{\mathbf{T}} = (\widehat{\tau}_{jk})_{d \times d}$ or $\mathbf{U}_n = \widehat{\mathbf{R}} = (\widehat{\rho}_{jk})_{d \times d}$ as in (2.6) respectively, and $\widehat{\boldsymbol{\Sigma}}$ the corresponding

estimator of Σ in (2.11) and (2.12). It follows from the expansion of the sine function in (2.11) and (2.12) that

$$(3.11) \quad (\widehat{\Sigma} - \Sigma)_{jk} \approx a_0(\mathbf{U}_n - \mathbb{E}\mathbf{U}_n)_{jk},$$

with $a_0 = \pi/2$ for $\mathbf{U}_n = \widehat{\mathbf{T}}$ and $a_0 = \pi/3$ for $\mathbf{U}_n = \widehat{\mathbf{R}}$. Thus, the estimators $\widehat{\Sigma}$ can be decomposed as

$$(3.12) \quad \begin{aligned} \widehat{\Sigma} - \Sigma &= a_0 \left\{ (\mathbf{U}_n - \mathbb{E}\mathbf{U}_n) - m\Delta_n^{(1)} \right\} + a_0 m \left(\Delta_n^{(1)} - \Delta_n^{(0)} \right) \\ &+ a_0 m \Delta_n^{(0)} + \left\{ (\widehat{\Sigma} - \Sigma) - a_0(\mathbf{U}_n - \mathbb{E}\mathbf{U}_n) \right\}, \end{aligned}$$

where the first two terms are bounded by (3.6) and (3.10) respectively and the third term is explicitly expressed as the difference between a sample covariance matrix and its expectation in (3.7). Moreover, the fourth term can be bounded with a higher order expansion of $\sin(t)$ in (2.11) and (2.12). We note that the fourth term on the right-hand side of (3.12) is not needed if one is interested in studying $\widehat{\mathbf{T}} - \mathbf{T}$ or $\widehat{\mathbf{R}} - \mathbf{R}$ without the sine transformation. This analysis leads to the following theorem.

THEOREM 1. *Let $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{R}}$ be respectively the Kendall's tau and Spearman's rho matrices in (2.6), \mathbf{T} and \mathbf{R} be their population version in (2.9), and $\widehat{\Sigma}^\tau = (\widehat{\Sigma}_{jk}^\tau)_{d \times d}$ and $\widehat{\Sigma}^\rho = (\widehat{\Sigma}_{jk}^\rho)_{d \times d}$ be the corresponding estimators in (2.11) and (2.12) for the population correlation matrix Σ in the Gaussian copula model (2.1). Then, for certain numerical constant C_0 and both $\widehat{\Sigma} = \widehat{\Sigma}^\tau$ and $\widehat{\Sigma} = \widehat{\Sigma}^\rho$*

$$(3.13) \quad \mathbb{E}\|\widehat{\Sigma} - \Sigma\|_S + \mathbb{E}\|\widehat{\mathbf{T}} - \mathbf{T}\|_S + \mathbb{E}\|\widehat{\mathbf{R}} - \mathbf{R}\|_S \leq C_0\|\Sigma\|_S \left(\sqrt{d/n} + d/n \right).$$

In particular, defining $n_2 = 2\lfloor n/2 \rfloor$ (where $\lfloor x \rfloor$ is the integer part of x),

$$(3.14) \quad \begin{aligned} \mathbb{E}\|\widehat{\mathbf{T}} - \mathbf{T}\|_S &\leq \sqrt{2d(d-2n)/\{n(n-1)\} + 4(2/\pi + 1)^2\|\Sigma\|_F^2/n} \\ &+ 10\|\Sigma\|_S \left(\sqrt{(d+1)/(3n)} + (d+1)/n \right), \end{aligned}$$

$$\mathbb{E}\|\widehat{\Sigma}^\tau - \Sigma\|_S \leq \frac{\pi}{2}\mathbb{E}\|\widehat{\mathbf{T}} - \mathbf{T}\|_S + \frac{\pi}{2}\sqrt{\frac{\|\Sigma\|_F^2 - d}{n_2}} + \frac{\pi^2\sqrt{3}d}{8n_2},$$

for Kendall's tau, and for Spearman's rho, with $n_3 = 3\lfloor n/3 \rfloor$

$$(3.15) \quad \begin{aligned} \mathbb{E}\|\widehat{\mathbf{R}} - \mathbf{R}\|_S &\leq \sqrt{6d(d-2n)/\{n(n-1)\} + 9(1 + \sqrt{8}/\pi)^2\|\Sigma\|_F^2/n} \\ &+ 15\|\Sigma\|_S \left(\sqrt{(d+1)/(3n)} + (d+1)/n \right) + \|\Sigma\|_F/n, \end{aligned}$$

$$\mathbb{E}\|\widehat{\Sigma}^\rho - \Sigma\|_S \leq \frac{\pi}{3}\mathbb{E}\|\widehat{\mathbf{R}} - \mathbf{R}\|_S + \frac{\pi}{9}\sqrt{\frac{\|\Sigma\|_F^2 - d}{n_3}} + \frac{\pi^2\sqrt{3}d}{36n_3} + \frac{2\pi\|\Sigma\|_F}{3n}.$$

COROLLARY 1. *If $\|\Sigma\|_S^2 d/n \rightarrow 0$, then*

$$\mathbb{E}\|\widehat{\mathbf{T}} - \mathbf{T}\|_S + \mathbb{E}\|\widehat{\Sigma}^\tau - \Sigma\|_S + \mathbb{E}\|\widehat{\mathbf{R}} - \mathbf{R}\|_S + \mathbb{E}\|\widehat{\Sigma}^\rho - \Sigma\|_S \rightarrow 0.$$

REMARK 1. Up to a numerical constant factor, Theorem 1 match the bound (1.1) for the expected spectral error of the oracle sample covariance matrix $\tilde{\Sigma}^s$. While Han and Liu (2013) and Wegkamp and Zhao (2013) focused on large deviation bound of the spectral error of $\|\hat{\Sigma}^\tau - \Sigma\|_S$ in the elliptical copula model, a direct application of their results requires $\|\Sigma\|_S d(\log d)/n \rightarrow 0$ for the convergence in spectrum norm. Although their results are of sharper order when $\|\Sigma\|_S \gg \log d$, it seems that when $\|\Sigma\|_S = \mathcal{O}(1)$, the extra logarithmic factor cannot be removed in their analysis based on the matrix Bernstein inequality (Tropp, 2011).

The proof of Theorem 1 requires a number of inequalities which provide key details of the analysis outlined above the statement of the theorem. These inequalities are crucial for our derivation of large deviation spectrum error bounds as well. We state these inequalities in a sequence of lemmas below and defer their proofs to the Appendix.

Let $\varphi_\rho(x, y)$ be the bivariate normal density with mean zero, variance one, and correlation ρ . Define

$$(3.16) \quad \bar{h}(x, y, \rho) = \int \int \text{sgn}(x - u) \text{sgn}(y - v) \varphi_\rho(u, v) du dv.$$

LEMMA 1. Let $\bar{h}(x, y, \rho)$ be as in (3.16). Based on $\mathbf{X} \in \mathbb{R}^{n \times d}$ with iid $N(0, \Sigma)$ rows, Kendall's $\hat{\tau}_{jk}$ is a U-statistic of order 2 with a permutation symmetric kernel $h_{j,k}(\mathbf{x}_1, \mathbf{x}_2)$ satisfying $|h_{j,k}(\mathbf{x}_1, \mathbf{x}_2)| = 1$ and

$$(3.17) \quad \mathbb{E}[h_{jk}(\mathbf{X}_1, \mathbf{X}_2) | \mathbf{X}_1 = \mathbf{x}] = \bar{h}(x_j, x_k, \Sigma_{jk}) \quad \forall j \neq k.$$

With $g(x, y, \rho) = \bar{h}(x, y, \rho) - \bar{h}(x, y, 0)$ and $C_1 = 2/\pi + 1$,

$$(3.18) \quad |g(x, y, \rho)| \leq C_1 |\rho|, \quad |(\partial/\partial x)g(x, y, \rho)| \leq |\rho|.$$

Moreover, with $\bar{h}_0(x) = 2\Phi(x) - 1$ and ρ_{jk} in (2.8),

$$(3.19) \quad \bar{h}(x, y, 0) = \bar{h}_0(x)\bar{h}_0(y), \quad \mathbb{E}\bar{h}(X_{ij}, X_{ik}, 0) = \rho_{jk}/3 \quad \forall j, k.$$

LEMMA 2. Let $\bar{h}(x, y, \rho)$ be as in (3.16) and $C_1 = \sqrt{8}/\pi + 1$. Based on $\mathbf{X} \in \mathbb{R}^{n \times d}$ with iid $N(\mathbf{0}, \Sigma)$ rows, Spearman's $\hat{\rho}_{jk}$ is a U-statistic of order 3 with a permutation symmetric kernel $h_{j,k}^\rho(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ satisfying

$$(3.20) \quad |\mathbb{E}\hat{\rho}_{jk} - \rho_{jk}| \leq |\rho_{jk}|/(n+1) \leq |\Sigma_{jk}|/(n+1),$$

$$(3.21) \quad |h_{jk}^\rho(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)| \leq 1,$$

$$(3.22) \quad |\bar{h}^\rho(x, y, \rho) - \bar{h}^\rho(x, y, 0)| \leq C_1 |\rho|,$$

$$(3.23) \quad |(\partial/\partial x)\{\bar{h}^\rho(x, y, \rho) - \bar{h}^\rho(x, y, 0)\}| \leq |\rho|,$$

$$(3.24) \quad (1 + 1/n)\bar{h}^\rho(x, y, 0) = \bar{h}(x, y, 0),$$

where $\bar{h}^\rho(x_j, x_k, \Sigma_{jk}) = \mathbb{E}[h_{jk}^\rho(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) | \mathbf{X}_1 = \mathbf{x}] - \tau_{jk}/(n+1)$.

LEMMA 3. Inequalities (3.6) and (3.10) hold with $C_1 = 2/\pi + 1 \leq 2$ and $m = 2$ for Kendall's tau and $C_1 \leq 1 + \sqrt{8}/\pi \leq 2$ and $m = 3$ for Spearman's rho. Moreover, for both Kendall's tau and Spearman's rho,

$$(3.25) \quad \mathbb{E} \|(\mathbf{U}_n - \mathbb{E}\mathbf{U}_n) - m\mathbf{\Delta}_n^{(0)}\|_F^2 \leq \frac{m(m-1)d(d-2n)_+}{n(n-1)} + C_1^2 \frac{\|\mathbf{\Sigma}\|_F^2}{n/m^2}.$$

LEMMA 4. Let $\mathbf{\Delta}_n^{(0)}$ as in (3.7) and $\mathbf{R} = (\rho_{jk})_{d \times d}$. Then,

$$(3.26) \quad \mathbb{E} \|\mathbf{\Delta}_n^{(0)}\|_S \leq 5\|\mathbf{\Sigma}\|_S \left(\sqrt{(d+1)/(3n)} + (d+1)/n \right)$$

and with at least probability $1 - 2e^{-t^2}$,

$$(3.27) \quad \|\mathbf{\Delta}_n^{(0)}\|_S \leq 5\|\mathbf{\Sigma}\|_S \left(\sqrt{(d+t^2/\pi)/(3n)} + (d+(t^2+1)/\pi)/n \right).$$

LEMMA 5. (i) Let $\widehat{\mathbf{\Sigma}}^\tau = (\widehat{\Sigma}_{jk}^\tau)_{d \times d}$ be as in (2.11) and $\mathbf{\Delta}^\tau = (\Delta_{jk}^\tau)_{d \times d}$ with $\Delta_{jk}^\tau = \widehat{\tau}_{jk} - \tau_{jk}$. Let $n_2 = 2\lfloor n/2 \rfloor$ where $\lfloor x \rfloor$ is the integer part of x . Then,

$$(3.28) \quad \sqrt{\mathbb{E} \|(\widehat{\mathbf{\Sigma}}^\tau - \mathbf{\Sigma}) - (\pi/2)\mathbf{\Delta}^\tau\|_F^2} \leq \frac{\pi}{2} \sqrt{\frac{\|\mathbf{\Sigma}\|_F^2 - d}{n_2}} + \frac{\pi^2 \sqrt{3}d}{8n_2}.$$

(ii) Let $\widehat{\mathbf{\Sigma}}^\rho = (\widehat{\Sigma}_{jk}^\rho)_{d \times d}$ be as in (2.12) and $\mathbf{\Delta}^\rho = (\Delta_{jk}^\rho)_{d \times d}$ with $\Delta_{jk}^\rho = \widehat{\rho}_{jk} - \mathbb{E}\widehat{\rho}_{jk}$. Let $n_3 = 3\lfloor n/3 \rfloor$ where $\lfloor x \rfloor$ is the integer part of x . Then,

$$(3.29) \quad \sqrt{\mathbb{E} \|(\widehat{\mathbf{\Sigma}}^\rho - \mathbf{\Sigma}) - (\pi/2)\mathbf{\Delta}^\rho\|_F^2} \leq \frac{\pi}{9} \sqrt{\frac{\|\mathbf{\Sigma}\|_F^2 - d}{n_3}} + \frac{\pi^2 \sqrt{3}d}{36n_3} + \frac{\pi \sqrt{\|\mathbf{\Sigma}\|_F^2 - d}}{3(n+1)}.$$

PROOF OF THEOREM 1. Let $n_m = m\lfloor n/m \rfloor$. As in (3.12), for Kendall's tau,

$$\begin{aligned} \|\widehat{\mathbf{T}} - \mathbf{T}\|_S &\leq \|(\mathbf{U}_n - \mathbb{E}\mathbf{U}_n) - 2\mathbf{\Delta}_n^{(0)}\|_F + 2\|\mathbf{\Delta}_n^{(0)}\|_S, \\ \|\widehat{\mathbf{\Sigma}}^\tau - \mathbf{\Sigma}\|_S &\leq \|(\widehat{\mathbf{\Sigma}}^\tau - \mathbf{\Sigma}) - (\pi/2)(\mathbf{U}_n - \mathbb{E}\mathbf{U}_n)\|_F + (\pi/2)\|\widehat{\mathbf{T}} - \mathbf{T}\|_S. \end{aligned}$$

with $\mathbf{U}_n = \widehat{\mathbf{T}}$ and $\mathbb{E}\mathbf{U}_n = \mathbf{T}$. It follows from (3.25) of Lemma 3 with $m = 2$, (3.26) of Lemma 4 and (3.28) of Lemma 5 that the inequalities in (3.14) hold.

Similarly, for Spearman's rho,

$$\begin{aligned} \|\widehat{\mathbf{R}} - \mathbf{R}\|_S &\leq \|(\mathbf{U}_n - \mathbb{E}\mathbf{U}_n) - 3\mathbf{\Delta}_n^{(0)}\|_F + 3\|\mathbf{\Delta}_n^{(0)}\|_S + \|\mathbb{E}\mathbf{U}_n - \mathbf{R}\|_F, \\ \|\widehat{\mathbf{\Sigma}}^\rho - \mathbf{\Sigma}\|_S &\leq \|(\widehat{\mathbf{\Sigma}}^\rho - \mathbf{\Sigma}) - (\pi/3)(\widehat{\mathbf{R}} - \mathbf{R})\|_F + (\pi/3)\|\widehat{\mathbf{R}} - \mathbf{R}\|_S, \end{aligned}$$

with $\mathbf{U}_n = \widehat{\mathbf{R}}$ and $\|\mathbb{E}\mathbf{U}_n - \mathbf{R}\|_F = \|(\mathbb{E}\widehat{\rho}_{jk} - \rho_{jk})_{d \times d}\|_F \leq \sqrt{\|\mathbf{\Sigma}\|_F^2 - d/(n+1)}$ by (3.20). Thus, (3.25), (3.26) and (3.29) yield the inequalities in (3.15). \blacksquare

4. Large Deviation Inequalities. While the upper bounds for the expected spectral error in Theorem 1 and Corollary 1 match (1.1) for the oracle sample covariance matrix, it is useful only when $d/n \rightarrow 0$ as is the case in many applications. For $d > n$, large deviation bounds for the sparse spectral norm of the form (1.2) is often used instead. In the present section we provide large deviation inequalities for both the spectral norm and the sparse spectral norm of the error for Kendall's tau and Spearman's rho.

The main result for this section is a large deviation bound in the following theorem for the maximum spectral error in a collection of diagonal submatrices.

THEOREM 2. *Let $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{R}}$ be respectively the Kendall's tau and Spearman's rho matrices in (2.6), \mathbf{T} and \mathbf{R} be their population version in (2.9), and $\widehat{\Sigma}^\tau = (\widehat{\Sigma}_{jk}^\tau)_{d \times d}$ and $\widehat{\Sigma}^\rho = (\widehat{\Sigma}_{jk}^\rho)_{d \times d}$ be the corresponding estimators in (2.11) and (2.12) for the population correlation matrix Σ in the Gaussian copula model (2.1). Let $1 \leq s \leq d$, $m \geq 1$ and $\mathcal{A}_{s,m}$ be a collection of m subsets $A \subset \{1, 2, \dots, d\}$ with $|A| \leq s$. Then, there exists a certain numerical constant C such that for both $\widehat{\Sigma} = \widehat{\Sigma}^\tau$ and $\widehat{\Sigma} = \widehat{\Sigma}^\rho$,*

$$(4.1) \quad \begin{aligned} & \|(\widehat{\Sigma} - \Sigma)_{A \times A}\|_S + \|(\widehat{\mathbf{T}} - \mathbf{T})_{A \times A}\|_S + \|(\widehat{\mathbf{R}} - \mathbf{R})_{A \times A}\|_S \\ & \leq C \|\Sigma_{A \times A}\|_S \left(\sqrt{(s+t+\log m)/n} + (s+t+\log m)/n \right) \\ & \quad + C \|\Sigma_{A \times A}\|_{(2,\infty)} \|\Sigma_{A \times A}\|_S^{1/2} \sqrt{(t+\log m)/n} + Cs(\log d + t)/n \end{aligned}$$

simultaneously for all $A \in \mathcal{A}_{s,m}$ with at least probability $1 - e^{-t}$.

COROLLARY 2. *If $t + \log d \leq \beta \max \{ \log(ed/s), \sqrt{(n/s)(t/s + \log(ed/s))} \}$, then for both $\widehat{\Sigma} = \widehat{\Sigma}^\tau$ and $\widehat{\Sigma} = \widehat{\Sigma}^\rho$ and a certain numerical constant C ,*

$$(4.2) \quad \begin{aligned} & \max_{|A| \leq s} \frac{\|(\widehat{\Sigma} - \Sigma)_{A \times A}\|_S + \mathbb{E}\|(\widehat{\mathbf{T}} - \mathbf{T})_{A \times A}\|_S + \mathbb{E}\|(\widehat{\mathbf{R}} - \mathbf{R})_{A \times A}\|_S}{\|\Sigma_{A \times A}\|_S + \|\Sigma_{A \times A}\|_S^{1/2} \|\Sigma_{A \times A}\|_{(2,\infty)}} \\ & \leq C(1 + \beta) \left(\sqrt{(t + s \log(ed/s))/n} + (t + s \log(ed/s))/n \right) \end{aligned}$$

with at least probability $1 - e^{-t}$.

REMARK 2. Corollary 2 illustrates that for $\max_{|A| \leq s} \|\Sigma_{A \times A}\|_S = \mathcal{O}(1)$ and under a mild condition on (n, d, s) , Theorem 2 yields a sparse spectral error bound that matches (1.2) of the latent $\widetilde{\Sigma}^s$. Note that $\|\Sigma_{A \times A}\|_{(2,\infty)} \leq \|\Sigma_{A \times A}\|_S$. In comparison, the spectral error bounds in Han and Liu (2013) and Wegkamp and Zhao (2013), which apply to the elliptical copula family, leads to $\max_{|A| \leq s} \|(\widehat{\Sigma}^\tau - \Sigma)_{A \times A}\|_S = \mathcal{O}(s\sqrt{(\log d)/n})$ by the union bound. Han and Liu (2013) provided a concentration inequality of order $\sqrt{s(\log d)/n}$ for $\widehat{\Sigma}^\tau$ in the transelliptical family under an additional 'sign sub-Gaussian' condition. They also provide two examples of elliptical copulas that satisfy the sign sub-Gaussian condition. The first example is the case of elliptical copulas with the latent correlation Σ satisfying a compound symmetric structure (i.e. $\Sigma_{jk} = \rho$ for all $j \neq k$).

The second example is the case when Σ has a diagonal block structure with each diagonal block having a compound symmetric structure. However, it is unclear if the sign sub-Gaussian condition is readily verifiable in general. Theorem 2 and Corollary 2 establish the concentration of the nonparametric estimates for the Gaussian copula model without the sign sub-Gaussianity condition, although the Gaussian copula family is smaller than the transelliptical family.

The corollary below states a simpler but slightly weaker version of Theorem 2 for $s = d$. It matches (1.2) for $s = d$ when $\|\Sigma\|_S = \mathcal{O}(1)$ and $t + \log d = \mathcal{O}(\sqrt{n/d})$.

COROLLARY 3. *For a certain numerical constant C ,*

$$(4.3) \quad \begin{aligned} \|\widehat{\Sigma} - \Sigma\|_S &\leq C\|\Sigma\|_S \left(\sqrt{(t+d)/n} + (t+d)/n \right) \\ &\quad + C\|\Sigma\|_S^{1/2} \|\Sigma\|_{(2,\infty)} \sqrt{t/n} + C(t + \log d)d/n \end{aligned}$$

with at least probability $1 - e^{-t}$ for both $\widehat{\Sigma} = \widehat{\Sigma}^\tau$ and $\widehat{\Sigma} = \widehat{\Sigma}^\rho$.

The proof of Theorem 2 is carried out by establishing large deviation inequalities for the first two terms in the decomposition in (3.12), an application of Lemma 4 to the third, and an application of an inequality of Wegkamp and Zhao (2013) to the fourth.

LEMMA 6. *Let us take $C_1 = 2/\pi + 1 \leq 2$ for Kendall's tau and $C_1 \leq 1 + \sqrt{8}/\pi \leq 2$ for Spearman's rho. For both Kendall's tau and Spearman's rho,*

$$(4.4) \quad \|\Delta_n^{(1)} - \Delta_n^{(0)}\|_S \leq \sqrt{\frac{C_1^2 \|\Sigma\|_F^2 - 2d}{n}} + 2\sqrt{2} \|\Sigma\|_{(2,\infty)} \|\Sigma\|_S^{1/2} \sqrt{\frac{t}{n}}$$

with at least probability $1 - e^{-t}$.

LEMMA 7. *Let $U_n - \mathbb{E}U_n - m\Delta_n^{(1)}$ be as in (3.12). Then, for a certain constant C ,*

$$\max_{|A| \leq s} \|(U_n - \mathbb{E}U_n - m\Delta_n^{(1)})_{A \times A}\|_S \leq Cs(\log d + t)/n$$

with at least probability $1 - e^{-t}$.

We state an inequality of Wegkamp and Zhao (2013) in Lemma 8 (i) below and its extension to Spearman's rho in Lemma 8 (ii).

LEMMA 8. (i) *Let $\widehat{\Sigma}^\tau = (\widehat{\Sigma}_{jk}^\tau)_{d \times d}$ be as in (2.11) and $\Delta^\tau = (\Delta_{jk}^\tau)_{d \times d}$ with $\Delta_{jk}^\tau = \widehat{\tau}_{jk} - \tau_{jk}$. Let $n_2 = 2\lfloor n/2 \rfloor$ where $\lfloor x \rfloor$ is the integer part of x . Then,*

$$(4.5) \quad \|(\widehat{\Sigma}^\tau - \Sigma)_{A \times A}\|_S \leq \pi \|(\widehat{T} - T)_{A \times A}\|_S + \frac{s\pi^2}{8} \|\Delta^\tau\|_{\max}^2,$$

with $\mathbb{P}\{\|\Delta^\tau\|_{\max} > 2t\} \leq d^2 e^{-n_2 t^2}$ for all $t > 0$.

(ii) Let $\widehat{\Sigma}^\rho = (\widehat{\Sigma}_{jk}^\rho)_{d \times d}$ be as in (2.12) and $\Delta^\rho = (\Delta_{jk}^\rho)_{d \times d}$ with $\Delta_{jk}^\rho = \widehat{\rho}_{jk} - \mathbb{E}\widehat{\rho}_{jk}$. Let $n_3 = 3\lfloor n/3 \rfloor$ where $\lfloor x \rfloor$ is the integer part of x . Then,

$$(4.6) \quad \|(\widehat{\Sigma}^\rho - \Sigma)_{A \times A}\|_S \leq C_2 \|(\widehat{\mathbf{T}} - \mathbf{T})_{A \times A}\|_S + \frac{s\pi^2}{36} \|\Delta^\rho\|_{\max}^2 + \frac{\pi s^{1/2} \|\Sigma_{A \times A}\|_{(2,\infty)}}{3(n+1)}$$

with $C_2 = (\pi/3)(2 - \sqrt{1-1/4}) < 1.2$, and $\mathbb{P}\{\|\Delta^\rho\|_{\max} > \sqrt{6}t\} \leq d^2 e^{-n_3 t^2}$ for all $t > 0$.

PROOF OF THEOREM 2. We consider only $\widehat{\Sigma}^\tau$ as the case for $\widehat{\Sigma}^\rho$ is nearly identical. It follows from Lemma 8 that

$$(4.7) \quad \|(\widehat{\Sigma}^\tau - \Sigma)_{A \times A}\|_S \leq \pi \|(\widehat{\mathbf{T}} - \mathbf{T})_{A \times A}\|_S + Cs(t + \log d)/n, \quad \forall |A| \leq s,$$

with at least probability $1 - e^{-t}$. As in the decomposition in (3.12),

$$(4.8) \quad \widehat{\mathbf{T}} - \mathbf{T} = \left\{ \Delta^\tau - 2\Delta_n^{(1)} \right\} + 2 \left\{ \Delta_n^{(1)} - \Delta_n^{(0)} \right\} + 2\Delta_n^{(0)}.$$

It follows from Lemma 7 that with at least probability $1 - e^{-t}$,

$$(4.9) \quad \max_{|A| \leq s} \left\| \left\{ \Delta^\tau - 2\Delta_n^{(1)} \right\}_{A \times A} \right\|_S \leq Cs(\log d + t)/n.$$

By applying Lemma 6 to the m sub-matrices with the union bound,

$$(4.10) \quad \begin{aligned} \|(\Delta_n^{(1)} - \Delta_n^{(0)})_{A \times A}\|_S &\leq C \|\Sigma_{A \times A}\|_F / \sqrt{n} \\ &+ C \|\Sigma_{A \times A}\|_{(2,\infty)} \|\Sigma_{A \times A}\|_S^{1/2} \sqrt{(t + \log m)/n}, \quad \forall A \in \mathcal{A}_{s,m}, \end{aligned}$$

with at least probability $1 - m \exp(-t - \log m) \geq 1 - e^{-t}$. Similarly, Lemma 4 yields

$$(4.11) \quad \begin{aligned} \|(\Delta_n^{(0)})_{A \times A}\|_S &\leq C \|\Sigma_{A \times A}\|_S \sqrt{(s + t + \log m)/n} \\ &+ C \|\Sigma_{A \times A}\|_S (s + t + \log m)/n, \quad \forall A \in \mathcal{A}_{s,m} \end{aligned}$$

with at least probability $1 - e^{-t}$. The first term in (4.10) is dominated by the first term in (4.11) due to $\|\Sigma_{A \times A}\|_F \leq \sqrt{s} \|\Sigma_{A \times A}\|_S$. Thus, applying (4.9), (4.10) and (4.11) to (4.8) yields (4.1) via (4.7). \blacksquare

5. Discussion. We describe two applications of our concentration inequality in the $d > n$ case.

5.1. *Tapering Estimate of Bandable Correlation Matrices.* We consider the Gaussian copula model in (2.1). We assume that the correlation matrix has a bandable structure in that the off-diagonal elements fall off to zero as we move further away from diagonal. There are several formulations of such bandability. As in Cai, Zhang and Zhou (2010), we consider the parameter class

$$(5.1) \quad \mathcal{F}_\alpha(M_0, M_1) = \left\{ \Sigma : \max_j \sum_{|i-j| > k} |\Sigma_{ij}| \leq M_0 k^{-\alpha} \forall k, \|\Sigma\|_S \leq M_1 \right\}.$$

We adopt the estimator of [Cai, Zhang and Zhou \(2010\)](#) and plug in $\widehat{\Sigma}^\tau$ and $\widehat{\Sigma}^\rho$:

$$(5.2) \quad \widehat{\Sigma}_{(k)}^{\tau-taper} = (w_{ij} \widehat{\Sigma}_{ij}^\tau)_{d \times d} \quad \widehat{\Sigma}_{(k)}^{\rho-taper} = (w_{ij} \widehat{\Sigma}_{ij}^\rho)_{d \times d}$$

where w_{ij} 's are defined as

$$w_{ij} = \begin{cases} 1 & \text{when } |i - j| \leq k/2 \\ 2 - 2\frac{|i - j|}{k} & \text{when } k/2 < |i - j| < k \\ 0 & \text{otherwise} \end{cases}$$

The nonparametric tapering estimator $\widehat{\Sigma}_{(k)}^{\rho-taper}$ has been considered previously in [Xue and Zou \(2012b\)](#), where an error bound

$$\sup_{\Sigma \in \mathcal{F}_\alpha(M_0, M_1)} \mathbb{E}_\Sigma \left\| \widehat{\Sigma}_{(k)}^{\rho-taper} - \Sigma \right\|_S^2 \leq C_{M_0, M_1} \left(\frac{k^2 \log d}{n} + k^{-2\alpha} \right)$$

was established using a generalization of McDiarmid's inequality, where \mathbb{E}_Σ is the expectation in the Gaussian copula model (2.1) with correlation Σ in (2.2), and C_{M_0, M_1} is a constant depending on M_0 and M_1 only. It was mentioned in their paper that the above error bound may not be sharp as some key concentration inequalities were not available for rank-based estimators. Such key concentration inequalities are provided in Theorem 2 as the rate-optimal error bound in the following theorem demonstrates.

THEOREM 3. *Let \mathbb{E}_Σ be the expectation under which (2.1) and (2.2) hold. Consider the tapered estimators $\widehat{\Sigma}_{(k)} = \widehat{\Sigma}_{(k)}^{\tau-taper}$ or $\widehat{\Sigma}_{(k)} = \widehat{\Sigma}_{(k)}^{\rho-taper}$ given in (5.2). Then,*

$$(5.3) \quad \sup_{\Sigma \in \mathcal{F}_\alpha(M_0, M_1)} \mathbb{E}_\Sigma \left\| \widehat{\Sigma}_{(k)} - \Sigma \right\|_S^2 \leq C_{M_0, M_1} \left(\frac{k + \log d}{n} + \frac{k^2 (\log d)^2}{n^2} + k^{-2\alpha} \right)$$

for all $1 \leq k \leq n$, where C_{M_0, M_1} is a constant depending on M_0 and M_1 only. In particular, for $k = \min(n^{1/(2\alpha+1)}, d)$ and $\log d \leq \beta n^{\alpha/(1+2\alpha)}$,

$$(5.4) \quad \sup_{\Sigma \in \mathcal{F}_\alpha(M_0, M_1)} \mathbb{E}_\Sigma \left\| \widehat{\Sigma}_{(k)} - \Sigma \right\|_S^2 \leq C_{M_0, M_1} (1 + \beta) \min \left(n^{\frac{-2\alpha}{1+2\alpha}} + \frac{\log d}{n}, \frac{d}{n} \right).$$

The rate-optimality of (5.4) was proved in [Cai, Zhang and Zhou \(2010\)](#) and a combination of their analysis and Theorem 2 proves Theorem 3. For $\mathbf{H} = (H_{ij})_{d \times d} = \widehat{\Sigma} - \Sigma$,

$$(w_{ij} H_{ij})_{d \times d} = k^{-1} \sum_{\ell=1}^{d+2k-1} \mathbf{H}_{A_\ell \times A_\ell} - k^{-1} \sum_{\ell=1}^{d+k-1} \mathbf{H}_{B_\ell \times B_\ell}$$

where $A_\ell = \{1 \vee (\ell - 2k), \dots, \ell\}$ for $1 \leq \ell < p + 2k$ and $B_\ell = \{1 \vee (\ell - k), \dots, \ell\}$ for $1 \leq \ell < p + k$. Let $A_{d+2k+\ell-1} = B_\ell$. Since $\{\mathbf{H}_{A_{\ell+2jk} \times A_{\ell+2jk}}, \ell + 2jk < d + 2k\}$

are disjoint diagonal blocks for $\ell = 1, \dots, 2k$ and $\{\mathbf{H}_{A_{\ell+jk} \times A_{\ell+jk}}, \ell + jk \geq d + 2k\}$ are disjoint diagonal blocks for $\ell = 1, \dots, k$,

$$\left\| (w_{ij} \widehat{\Sigma}_{ij})_{d \times d} - \Sigma \right\|_S \leq \left\| ((1 - w_{ij}) \Sigma_{ij})_{d \times d} \right\|_S + 3 \max_{\ell \leq 2d+3k-2} \left\| \mathbf{H}_{A_\ell \times A_\ell} \right\|_S$$

with $|A_\ell| \leq 2k$. Since $w_{ij} = 0$ for $|i - j| \leq k$, the first term above is bounded by $M_0 k^{-\alpha}$ in the class. It follows from Theorem 2 that the second term above is bounded by

$$\mathbb{E}_\Sigma \max_{\ell \leq 2d+3k-2} \left\| \mathbf{H}_{A_\ell \times A_\ell} \right\|_S^2 \leq C_{M_0, M_1} \int_0^\infty \left(\frac{k + t + \log d}{n} + \frac{k^2 (\log d + t)^2}{n^2} \right) e^{-t} dt,$$

which implies (5.3).

Although the estimator in (5.2) is not adaptive due to the requirement of k as an input, this example demonstrates the utility of our results when Kendall's tau and Spearman's rho are used in place of the oracle sample covariance matrix. Based on the availability of the latent sample covariance matrix $\widehat{\Sigma}^s$, Cai and Yuan (2012) proposed a block thresholding estimator to achieve the optimal rate in (5.4) without the knowledge of α . An interesting problem is whether the same can be achieved using the Kendall's tau or Spearman's rho, as it seems to need a modification of Theorem 2 for off diagonal blocks of the error $\widehat{\Sigma} - \Sigma$.

5.2. Principal Component Analysis. Theorem 1 immediately yields the following theorem via the Weyl (1912) and Davis and Kahan (1970) inequalities.

THEOREM 4. *Consider the Gaussian copula model in (2.1). Let \mathbf{P}_k , $\widehat{\mathbf{P}}_k^\tau$ and $\widehat{\mathbf{P}}_k^\rho$ be the projections to the span of the k leading eigenvectors of Σ , $\widehat{\Sigma}^\tau$ and $\widehat{\Sigma}^\rho$ respectively corresponding to their k largest eigenvalues. Let λ_j be the j -th largest eigenvalue of Σ . Then, for a certain numerical constant C ,*

$$\max \left(\mathbb{E} \left\| \widehat{\mathbf{P}}_k^\tau - \mathbf{P}_k \right\|_S, \mathbb{E} \left\| \widehat{\mathbf{P}}_k^\rho - \mathbf{P}_k \right\|_S \right) \leq C \|\Sigma\|_S (\sqrt{d/n} + d/n) / (\lambda_k - \lambda_{k+1}).$$

Now we consider the problem of estimating the direction of a sparse leading eigenvector. We illustrate the utility of our sparse spectral error bound in the sparse PCA problem by plugging in $\{\widehat{\Sigma}^\tau, \widehat{\Sigma}^\rho\}$ in place of $\widehat{\Sigma}^s$ in a formulation of Vu and Lei (2012). In particular, we consider an integer $s < d$ to be an upper bound on the number of nonzero components of the principal eigenvector θ_1 of Σ . The following describes the sparse estimates of the principal eigenvector based on $\widehat{\Sigma}^\tau$ and $\widehat{\Sigma}^\rho$.

$$(5.5) \quad \widehat{\theta}_{1;s}^\tau = \arg \max_{\mathbf{v} \in \mathbb{S}^{d-1}: \|\mathbf{v}\|_0 \leq s} \left| \mathbf{v}^T \widehat{\Sigma}^\tau \mathbf{v} \right| \quad \widehat{\theta}_{1;s}^\rho = \arg \max_{\mathbf{v} \in \mathbb{S}^{d-1}: \|\mathbf{v}\|_0 \leq s} \left| \mathbf{v}^T \widehat{\Sigma}^\rho \mathbf{v} \right|$$

The following theorem provides the rate of convergence for sparse PCA.

THEOREM 5 (Sparse PCA). *Consider the Gaussian copula model in (2.1). Let (λ_1, θ_1) be the leading eigenpair of Σ with $\|\theta_1\|_0 \leq s \rightarrow \infty$. Let λ_2 be the second*

largest eigenvalue of Σ . Let $\hat{\theta}_{1;s}^\tau$ and $\hat{\theta}_{1;s}^\rho$ be the estimate obtained by the optimization defined in (5.5). If $t + \log d \leq \beta \sqrt{(n/s)(t + \log(ed/s))}$, then for both $\hat{\theta}_{1;s} = \hat{\theta}_{1;s}^\tau$ and $\hat{\theta}_{1;s} = \hat{\theta}_{1;s}^\rho$ and some numeric constant $C > 0$,

$$\left| \sin \angle(\hat{\theta}_{1;s}, \theta_1) \right| \leq \frac{C(1+\beta)}{\lambda_1 - \lambda_2} \left(\|\Sigma\|_S + \|\Sigma\|_S^{1/2} \|\Sigma\|_{(2,\infty)} \right) \sqrt{(t + s \log(ed/s))/n}$$

with probability at least $1 - e^{-t}$.

Theorem 5 follows from Corollary 2 by an application of a similar result from Wang, Han and Liu (2013). We omit the proofs.

APPENDIX A: AUXILIARY LEMMAS

PROOF OF LEMMA 1. By (2.4), the kernel for Kendall's tau is

$$h_{j,k}(\mathbf{x}_1, \mathbf{x}_2) = \text{sgn}(x_{1j} - x_{2j}) \text{sgn}(x_{1k} - x_{2k}).$$

The definition of $\bar{h}(x, y, \rho)$ in (3.16) directly yields (3.17) and the first identity of (3.19). It remains to verify the properties of $g(x, y, \rho)$ in (3.18) and compute the expectation in (3.19).

We first prove the following inequality:

$$(A.1) \quad \max_y \left| \Phi(y) - \Phi(y\sqrt{1-\rho^2}) \right| \leq |\rho|/2, \quad \forall -1 \leq \rho \leq 1.$$

For fixed ρ , the above maximum is attained, $(d/dy)\{\Phi(y) - \Phi(y\sqrt{1-\rho^2})\} = 0$, when $e^{-y^2/2} = \sqrt{1-\rho^2}e^{-y^2(1-\rho^2)/2}$ or equivalently $(1-\rho^2)e^{y^2\rho^2} = 1$. Let $y_\rho = \rho^{-1}\sqrt{-\log(1-\rho^2)}$ be the solution. Since the equality is attained in (A.1) at $\rho = 1$, (A.1) is a consequence of

$$(A.2) \quad \begin{aligned} & \frac{d}{d\rho} \frac{\Phi(y_\rho) - \Phi(y_\rho\sqrt{1-\rho^2})}{\rho} \\ &= \frac{\varphi(y_\rho\sqrt{1-\rho^2})}{\sqrt{1-\rho^2}} - \frac{\Phi(y_\rho) - \Phi(y_\rho\sqrt{1-\rho^2})}{\rho^2} \\ &\geq 0. \end{aligned}$$

By the monotonicity of the normal density $\varphi(t)$ in $|t|$,

$$\Phi(y_\rho) - \Phi(y_\rho\sqrt{1-\rho^2}) \leq y_\rho(1 - \sqrt{1-\rho^2})\varphi(y_\rho\sqrt{1-\rho^2}).$$

Since $y_\rho\rho = \sqrt{-\log(1-\rho^2)} \leq \sqrt{\rho^2/(1-\rho^2)}$, (A.2) follows from

$$y_\rho(1 - \sqrt{1-\rho^2}) = \frac{y_\rho\rho^2}{1 + \sqrt{1-\rho^2}} \leq \frac{\rho^2}{\sqrt{1-\rho^2}}.$$

This completes the proof of (A.1).

The joint normal density can be factorized as $\varphi_\rho(u, v) = \varphi(u)\varphi_\rho(v|u)$ with the conditional density $\varphi_\rho(v|u) \sim N(\rho u, 1 - \rho^2)$. By (3.16),

$$\begin{aligned}
 g(x, y, \rho) &= \int \operatorname{sgn}(x - u)\varphi(u) \left\{ \int \operatorname{sgn}(y - v) \{ \varphi_\rho(v|u) - \varphi(v) \} dv \right\} du \\
 &= \int \operatorname{sgn}(x - u)\varphi(u) \left\{ 2 \int_{-\infty}^y \{ \varphi_\rho(v|u) - \varphi(v) \} dv \right\} du \\
 (A.3) \quad &= 2 \int \operatorname{sgn}(x - u)\varphi(u) \left\{ \Phi((y - \rho u)/\sqrt{1 - \rho^2}) - \Phi(y) \right\} du.
 \end{aligned}$$

This gives the first part of (3.18) since $|\Phi((y - \rho u)/\sqrt{1 - \rho^2}) - \Phi(y - \rho u)| \leq |\rho|/2$ by (A.1) and $|\Phi(y - \rho u) - \Phi(y)| \leq |\rho u|/\sqrt{2\pi}$.

Similarly, since $\operatorname{sgn}(x - u) = 2I\{u \leq x\} - 1$,

$$\begin{aligned}
 \frac{\partial}{\partial x} g(x, y, \rho) &= \frac{\partial}{\partial x} 4 \int_{-\infty}^x \varphi(u) \left\{ \Phi((y - \rho u)/\sqrt{1 - \rho^2}) - \Phi(y) \right\} du \\
 &= 4\varphi(x) \left\{ \Phi((y - \rho x)/\sqrt{1 - \rho^2}) - \Phi(y) \right\}.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \left| \frac{\partial}{\partial x} g(x, y, \rho) \right| &= 4\varphi(x) \left| \Phi((y - \rho x)/\sqrt{1 - \rho^2}) - \Phi(y) \right| \\
 &\leq 4\varphi(x) \left(\frac{|\rho x|}{\sqrt{2\pi}} + \frac{|\rho|}{2} \right).
 \end{aligned}$$

This gives the second part of (3.18) due to

$$\max_{x > 0} 4\varphi(x)(x/\sqrt{2\pi} + 1/2) \leq 0.987 < 1.$$

For $j \neq k$, (2.8) gives

$$\mathbb{E}\bar{h}_0(X_{1j}, X_{1k}, 0) = \mathbb{E}\operatorname{sgn}(X_{1j} - X_{2j})\operatorname{sgn}(X_{1k} - X_{3k}) = \rho_{jk}/3.$$

Since $U = \Phi(X_1) \sim \text{uniform}(0, 1)$, $\int \bar{h}_0^2(x)\varphi(x)dx = 4\operatorname{Var}(U) = 1/3$. The second identity of (3.19) follows. \blacksquare

PROOF OF LEMMA 2. We need to include the sample size n in the subscript. As in Hoeffding (1948), Spearman's rho can be written as

$$(A.4) \quad \hat{\rho}_{n,jk} = \frac{n-2}{n+1} u_{n,jk} + \frac{3}{n+1} \hat{\tau}_{n,jk}$$

where $u_{n,jk}$ is a U-statistic of order 3 with kernel

$$(A.5) \quad h_{jk}^*(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = 3\operatorname{sgn}(x_{1j} - x_{2j})\operatorname{sgn}(x_{1k} - x_{3k}).$$

For $x \in [0, \pi/2]$, both $\sin x$ and $\sin x - 2\sin(x/3)$ are concave functions with $\sin x - 2\sin(x/3) = 0$ at the two endpoints, so that $\sin(2x/3) \leq 2\sin(x/3) \leq \sin x$. Thus, with $x = \pi|\rho_{jk}|/2$, (2.10) implies that

$$(A.6) \quad \text{sgn}(\tau_{jk}) = \text{sgn}(\rho_{jk}), \quad (\pi/3)|\rho_{jk}| \leq (\pi/2)|\tau_{jk}| \leq (\pi/2)|\rho_{jk}|.$$

Since $\mathbb{E}u_{jk} = \rho_{jk}$, $|\mathbb{E}\hat{\rho}_{jk} - \rho_{jk}| = 3|\rho_{jk} - \tau_{jk}|/(n+1) \leq |\rho_{jk}|/(n+1)$. This gives (3.20) as $|\rho_{jk}| \leq |\Sigma_{jk}|$ by the concavity of $\sin(t)$ in $(0, \pi/6)$. Since $u_{n,jk}$ and $\hat{\tau}_{n,jk}$ are U-statistics with kernel independent of n , $\hat{\rho}_{n,jk}$ is a U-statistic with kernel

$$(A.7) \quad h_{jk}^\rho(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) = \frac{n-2}{n+1}u_{3,jk} + \frac{3}{n+1}\hat{\tau}_{3,jk}.$$

Since $|u_{3,jk}| = |4\hat{\rho}_{3,jk} - 3\hat{\tau}_{3,jk}| \leq 1$ always holds, (3.21) follows.

Let $\bar{g}(x, \rho) = \int \bar{h}(x, y, \rho)\varphi(y)dy$. It follows from (A.5) that

$$\mathbb{E}[u_{3,jk} | \mathbf{X}_1 = \mathbf{x}] = \bar{h}(x_j, x_k, 0) + \bar{g}(x_j, \Sigma_{jk}) + \bar{g}(x_k, \Sigma_{jk}).$$

Similarly, $\mathbb{E}[3\hat{\tau}_{3,jk} | \mathbf{X}_1 = \mathbf{x}] = 2\bar{h}(x_j, x_k, \Sigma_{jk}) + \tau_{jk}$. Thus, we may take

$$\begin{aligned} \bar{h}^\rho(x_j, x_k, \Sigma_{jk}) &= \frac{n-2}{n+1} \left(\bar{h}(x_j, x_k, 0) + \bar{g}(x_j, \Sigma_{jk}) + \bar{g}(x_k, \Sigma_{jk}) \right) \\ &\quad + \frac{2}{n+1} \bar{h}(x_j, x_k, \Sigma_{jk}) \end{aligned}$$

with $c_{jk} = \tau_{jk}/(n+1)$ in (3.3). Since $\bar{g}(x, 0) = \int \bar{h}(x, 0)\bar{h}(y, 0)\varphi(y)dy = 0$, (3.24) holds. Moreover, with $g(x, y, \rho) = \bar{h}(x, y, \rho) - \bar{h}(x, y, 0)$ as in (3.18),

$$\bar{h}^\rho(x, y, \rho) - \bar{h}^\rho(x, y, 0) = \frac{n-2}{n+1} \left(\bar{g}(x, \rho) + \bar{g}(y, \rho) \right) + \frac{2}{n+1} g(x, y, \rho),$$

so that (3.22) and (3.23) are consequences of

$$(A.8) \quad |\bar{g}(x, \rho)| \leq |\rho| \left(\frac{\sqrt{2}}{\pi} + \frac{1}{2} \right), \quad \left| \frac{\partial}{\partial x} \bar{g}(x, \rho) \right| \leq |\rho|,$$

Since $\int \text{sgn}(x-u)\varphi(x)dx = -\bar{h}_0(u)$, (A.3) and (A.1) yield

$$\begin{aligned} |\bar{g}(y, \rho)| &= \left| 2 \int \bar{h}_0(u)\varphi(u) \left\{ \Phi((y-\rho u)/\sqrt{1-\rho^2}) - \Phi(y) \right\} du \right| \\ &\leq 2 \int |\bar{h}_0(u)(\rho/2 + \rho u/\sqrt{2\pi})| \varphi(u) du \end{aligned}$$

Since $\int |\bar{h}_0(u)|\varphi(u)du = \int_0^1 |2x-1|dx = 1/2$ and

$$\int \bar{h}_0(u)u\varphi(u)du = -2 \int_0^\infty \bar{h}_0(u)d\varphi(u) = 2 \int \varphi^2(u)du = 1/\sqrt{\pi},$$

we have $|\bar{g}(y, \rho)| \leq |\rho|(1/2 + \sqrt{2}/\pi)$. In addition, (3.18) yields

$$\left| \frac{\partial}{\partial x} \bar{g}(x, \rho) \right| \leq \max_{x,y} \left| \frac{\partial}{\partial x} g(x, y, \rho) \right| \leq |\rho|.$$

Hence, (A.8) holds and the proof is complete. ■

PROOF OF LEMMA 3. By Lemmas 1 and 2, both Kendall's tau and Spearman's rho are U-statistics with kernel bounded by 1, so that (3.6) holds. By (3.18) and (3.22), (3.9) holds, so that (3.10) holds. Since completely degenerate U-statistics of order two or higher are orthogonal to U-statistics of order 1, (3.6) and (3.10) yield

$$\begin{aligned} & \mathbb{E} \|(\mathbf{U}_n - \mathbb{E}\mathbf{U}_n) - m\Delta_n^{(0)}\|_F^2 \\ & \leq \frac{m(m-1)d(d-1)}{n(n-1)} + m^2 \left(C_1^2 \sum_{j \neq k} \frac{\Sigma_{jk}^2}{n} + \frac{4d}{45n} \right). \end{aligned}$$

Inequality (3.25) follows from $C_1^2 \geq 2 + 4/45$ and $\sum_{j \neq k} \Sigma_{jk}^2 = \|\Sigma\|_F^2 - d$. \blacksquare

PROOF OF LEMMA 4. Let N_ϵ be the largest number of ϵ -balls one can pack in the $(1 + \epsilon)$ -ball centered at the origin and $\{\mathbf{u}_{(j)}, j \leq N_\epsilon\}$ be the centers of such ϵ -balls in one of such configurations. From straight forward volume comparison we have $N_\epsilon \epsilon^d \leq (1 + \epsilon)^d$. For each $\mathbf{u} \in \mathbb{S}^{d-1}$, $\|\mathbf{u} - \mathbf{u}_{(j)}\|_2 \leq 2\epsilon$ for some $j \leq N_\epsilon$, so that

$$\begin{aligned} \left| \mathbf{u}^T \Delta_n^{(0)} \mathbf{u} \right| & \leq \left| \mathbf{u}_{(j)}^T \Delta_n^{(0)} \mathbf{u}_{(j)} \right| + \left| (\mathbf{u} - \mathbf{u}_{(j)})^T \Delta_n^{(0)} (\mathbf{u} + \mathbf{u}_{(j)}) \right| \\ & \leq \left| \mathbf{u}_{(j)}^T \Delta_n^{(0)} \mathbf{u}_{(j)} \right| + 2\epsilon(2 + 2\epsilon) \|\Delta_n^{(0)}\|_S. \end{aligned}$$

It follows that

$$(A.9) \quad \|\Delta_n^{(0)}\|_S \leq \sup_{j \leq N_\epsilon} \frac{|\mathbf{u}_{(j)}^T \Delta_n^{(0)} \mathbf{u}_{(j)}|}{1 - 4\epsilon(1 + \epsilon)}, \quad N_\epsilon \leq (1 + 1/\epsilon)^d.$$

Since \mathbf{X} has iid $N(0, \Sigma)$ rows, it can be written as $\mathbf{X} = \mathbf{Z}\Sigma^{1/2}$ with a standard normal matrix $\mathbf{Z} \in \mathbb{R}^{n \times d}$. Let $\bar{h}_0(\mathbf{X})$ be the $n \times d$ matrix with elements $\bar{h}_0(X_{ij}) = 2\Phi(X_{ij}) - 1$ and

$$f_{\mathbf{u}}(\mathbf{Z}) = \|\bar{h}_0(\mathbf{Z}\Sigma^{1/2})\mathbf{u}\|_2 / \sqrt{n}.$$

By (3.7), $\Delta_n^{(0)}$ has elements $(\mathbb{E}_n - \mathbb{E})\bar{h}_0(x_j)\bar{h}_0(x_k)$ so that

$$(A.10) \quad \mathbf{u}^T \Delta_n^{(0)} \mathbf{u} = f_{\mathbf{u}}^2(\mathbf{Z}) - \mathbb{E} f_{\mathbf{u}}^2(\mathbf{Z}).$$

Since $(d/dt)\Phi(t) \leq 1/\sqrt{2\pi}$, for any $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times d}$ we have

$$|f_{\mathbf{u}}(\mathbf{V}) - f_{\mathbf{u}}(\mathbf{W})| \leq \sqrt{\frac{2}{n\pi}} \|(\mathbf{V} - \mathbf{W})\Sigma^{1/2}\|_F \leq \sqrt{\frac{2\|\Sigma\|_S}{n\pi}} \|\mathbf{V} - \mathbf{W}\|_F$$

Thus, the Lipschitz norm of $f_{\mathbf{u}}(\cdot)$ is bounded by $\sqrt{2\|\Sigma\|_S/(n\pi)}$. By the Gaussian concentration inequality (Borell, 1975), we have

$$(A.11) \quad \mathbb{P}\left\{ \left| f_{\mathbf{u}}(\mathbf{Z}) - \mathbb{E} f_{\mathbf{u}}(\mathbf{Z}) \right| > t \sqrt{2\|\Sigma\|_S/(\pi n)} \right\} \leq 2e^{-t^2/2}.$$

It follows that

$$\mathbb{E}f_{\mathbf{u}}^2(\mathbf{Z}) - \left(\mathbb{E}f_{\mathbf{u}}(\mathbf{Z})\right)^2 = \text{Var}\left(f_{\mathbf{u}}(\mathbf{X})\right) \leq \frac{2\|\boldsymbol{\Sigma}\|_S}{\pi n} \int_0^\infty e^{-t^2/2} dt^2 = \frac{4\|\boldsymbol{\Sigma}\|_S}{\pi n}.$$

We note that $\mathbb{E}f_{\mathbf{u}}^2(\mathbf{Z}) = \mathbf{u}^T \mathbf{R} \mathbf{u} / 3 \leq \|\mathbf{R}\|_S / 3$ as in (3.8), so that by (A.10)

$$\begin{aligned} |\mathbf{u}^T \boldsymbol{\Delta}_n^{(0)} \mathbf{u}| &\leq \left| f_{\mathbf{u}}^2(\mathbf{X}) - \left(\mathbb{E}f_{\mathbf{u}}(\mathbf{X})\right)^2 \right| + \frac{4\|\boldsymbol{\Sigma}\|_S}{\pi n} \\ &\leq \left(f_{\mathbf{u}}(\mathbf{X}) - \mathbb{E}f_{\mathbf{u}}(\mathbf{X}) \right)^2 + 2 \left(\|\mathbf{R}\|_S / 3 \right)^{1/2} \left| f_{\mathbf{u}}(\mathbf{X}) - \mathbb{E}f_{\mathbf{u}}(\mathbf{X}) \right| + \frac{4\|\boldsymbol{\Sigma}\|_S}{\pi n}. \end{aligned}$$

This inequality and (A.9) yield

$$(A.12) \quad \|\boldsymbol{\Delta}_n^{(0)}\|_S \leq \frac{\zeta_n^2 + 2 \left(\|\mathbf{R}\|_S / 3 \right)^{1/2} \zeta_n + 4\|\boldsymbol{\Sigma}\|_S / (\pi n)}{1 - 4\epsilon(1 + \epsilon)}$$

with $\zeta_n = \max_{j \leq (1+1/\epsilon)^d} \left| f_{\mathbf{u}_{(j)}}(\mathbf{X}) - \mathbb{E}f_{\mathbf{u}_{(j)}}(\mathbf{X}) \right|$. It follows from (A.11) that

$$(A.13) \quad \mathbb{P}\left\{ \zeta_n > t \sqrt{2\|\boldsymbol{\Sigma}\|_S / (\pi n)} \right\} \leq 2(1 + 1/\epsilon)^d e^{-t^2/2}.$$

Let $x_* = 2(d \log(1 + 1/\epsilon) + \log 2)$. We have

$$\mathbb{E}\zeta_n^2 \leq \frac{2\|\boldsymbol{\Sigma}\|_S}{\pi n} \int_0^\infty \min \left\{ 2(1 + 1/\epsilon)^d e^{-t^2/2}, 1 \right\} dt^2 = \frac{2\|\boldsymbol{\Sigma}\|_S}{\pi n} (x_* + 2)$$

Taking ϵ satisfying $\epsilon(1 + \epsilon) = 1/20$, we find $1/(1 - 4\epsilon(1 + \epsilon)) = 5/4$ and $\log(1 + 1/\epsilon) \leq \pi$, so that $x_* \leq 2(\pi d + \log 2)$ and

$$\mathbb{E}\zeta_n^2 \leq 4\|\boldsymbol{\Sigma}\|_S (d/n + (1 + \log 2)/(\pi n)).$$

Combining this with (A.12), we have

$$\begin{aligned} \mathbb{E}\|\boldsymbol{\Delta}_n^{(0)}\|_S &\leq (5/4) \left\{ \mathbb{E}\zeta_n^2 + 2(\|\mathbf{R}\|_S / 3)^{1/2} \mathbb{E}\zeta_n + 4\|\boldsymbol{\Sigma}\|_S / (\pi n) \right\} \\ &\leq 5\|\boldsymbol{\Sigma}\|_S \left\{ d/n + (2 + \log 2)/(\pi n) \right\} \\ &\quad + 5(\|\boldsymbol{\Sigma}\|_S \|\mathbf{R}\|_S / 3)^{1/2} (d/n + (1 + \log 2)/(\pi n))^{1/2}. \end{aligned}$$

This yields (3.26) due to $2 + \log 2 \leq \pi$ and $\|\mathbf{R}\|_S \leq \|\boldsymbol{\Sigma}\|_S$. Moreover, by (A.13)

$$\mathbb{P}\left\{ \zeta_n > \sqrt{2\pi d + 2t^2} \sqrt{2\|\boldsymbol{\Sigma}\|_S / (\pi n)} \right\} \leq 2e^{\pi d - (2\pi d + 2t^2)/2} = 2e^{-t^2}$$

and outside this event (A.12) gives

$$\begin{aligned} \|\boldsymbol{\Delta}_n^{(0)}\|_S &\leq 5\|\boldsymbol{\Sigma}\|_S (d/n + (t^2 + 1)/(\pi n)) \\ &\quad + 5(\|\boldsymbol{\Sigma}\|_S \|\mathbf{R}\|_S / 3)^{1/2} \sqrt{d/n + t^2/(\pi n)}. \end{aligned}$$

This completes the proof due to $\|\mathbf{R}\|_S \leq \|\boldsymbol{\Sigma}\|_S$. ■

PROOF OF LEMMA 5. (i) Let $x = (\pi/2)\tau_{jk}$ and $y = (\pi/2)\Delta_{jk}^\tau$ so that $\widehat{\Sigma}_{jk} = \sin(x + y)$ and $\Sigma_{jk} = \sin x$. Because $\sin(x + y) - \sin x - y = (\cos x - 1)y - \int_0^y (y - t) \sin(x + t) dt$,

$$\left| \widehat{\Sigma}_{jk}^\tau - \Sigma_{jk} - (\pi/2)\Delta_{jk}^\tau \right| \leq \frac{2|xy|}{\pi} + \frac{y^2}{2} \leq \frac{\pi}{2} |\tau_{jk}\Delta_{jk}^\tau| + \frac{\pi^2}{8} |\Delta_{jk}^\tau|^2.$$

Since $\widehat{\tau}_{jk}$ is a U-statistic of order $m = 2$ and a sign kernel in (2.4), the Hoeffding decoupling argument gives $\mathbb{E}(\Delta_{jk}^\tau)^2 \leq \mathbb{E}(2 \text{Bin}(n_2, p_{jk})/n_2 - 2p_{jk})^2 \leq 1/n_2$ and

$$\mathbb{E}(\Delta_{jk}^\tau)^4 \leq \mathbb{E}\left(2 \text{Bin}(n_2, p_{jk})/n_2 - 2p_{jk}\right)^4 \leq 3/n_2^2,$$

where $p_{jk} = (1 + \tau_{jk})/2$. Since $\sum_{j \neq k} \tau_{jk}^2 \leq \sum_{j \neq k} \Sigma_{jk}^2 = \|\Sigma\|_F^2 - d$, we have

$$\sum_{j,k} \mathbb{E} |\tau_{jk}\Delta_{jk}^\tau|^2 \leq \frac{\|\Sigma\|_F^2 - d}{n_2}, \quad \sum_{j,k} \mathbb{E} |\Delta_{jk}^\tau|^4 \leq \frac{3d^2}{n_2^2}.$$

Consequently, (3.28) holds.

(ii) Let $x = (\pi/6)\mathbb{E}\widehat{\rho}_{jk}$, $y = (\pi/6)\Delta_{jk}^\rho$ and $z = (\pi/6)(\mathbb{E}\widehat{\rho}_{jk} - \rho_{jk})$ so that $\widehat{\Sigma}_{jk} = 2\sin(x + y)$ and $\Sigma_{jk} = 2\sin(x - z)$. Due to $|z| \leq (\pi/6)|\Sigma_{jk}|/(n+1)$ by (3.20),

$$\begin{aligned} \left| \widehat{\Sigma}_{jk}^\rho - \Sigma_{jk} - \frac{\pi}{3}\Delta_{jk}^\rho \right| &= 2 \left| \sin(x + y) - \sin(x - z) - y \right| \\ &\leq \frac{4|xy|}{\pi} + y^2 + 2|z| \\ &\leq \frac{\pi}{9} |\Sigma_{jk}\Delta_{jk}^\rho| + \frac{\pi^2}{36} |\Delta_{jk}^\rho|^2 + \frac{\pi|\Sigma_{jk}|}{3(n+1)}. \end{aligned}$$

Similar to part (i), (3.29) follows from $\mathbb{E}(\Delta_{jk}^\tau)^2 \leq 1/n_3$ and $\mathbb{E}(\Delta_{jk}^\tau)^4 \leq 3/n_3^2$. ■

PROOF OF LEMMA 6. We write

$$\Delta_n^{(1)} - \Delta_n^{(0)} = (\mathbb{E}_n - \mathbb{E})\mathbf{G} = n^{-1} \sum_{i=1}^n \mathbf{G}(\mathbf{X}_i) - \mathbb{E}\mathbf{G}(\mathbf{X}_1)$$

with $\mathbf{G}(\mathbf{x}) = (g_{jk}(\mathbf{x}))_{d \times d}$, where $g_{jk}(\mathbf{x}) = \bar{h}^\rho(x_j, x_k, \Sigma_{jk}) - \bar{h}^\rho(x_j, x_k, 0)$ for Kendall's tau and $g_{jk}(\mathbf{x}) = \bar{h}(x_j, x_k, \Sigma_{jk}) - \bar{h}(x_j, x_k, 0)$ for Spearman's rho. It follows from (3.18) and (3.23) that

$$|g_{jk}(\mathbf{y}) - g_{jk}(\mathbf{x})| \leq |\Sigma_{jk}| \{|y_j - x_j| + |y_k - x_k|\}.$$

This inequality implies that for all d -dimensional vectors \mathbf{x} and \mathbf{y} ,

$$\begin{aligned}
\|\mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{y})\|_S &\leq \max_{\mathbf{u}: \|\mathbf{u}\|_2=1} \sum_{j=1}^d \sum_{k=1}^d |u_j u_k| |g_{jk}(\mathbf{x}) - g_{jk}(\mathbf{y})| \\
&\leq \max_{\mathbf{u}: \|\mathbf{u}\|_2=1} \sum_{j=1}^d \sum_{k=1}^d |u_j u_k \Sigma_{jk}| (|x_j - y_j| + |x_k - y_k|) \\
&\leq 2 \max_{\mathbf{u}: \|\mathbf{u}\|_2=1} \sum_{j=1}^d \sum_{k=1}^d |u_j u_k \Sigma_{jk}| (x_j - y_j) \\
&\leq 2 \max_{\mathbf{u}: \|\mathbf{u}\|_2=1} \sum_{j=1}^d |u_j (x_j - y_j)| \max_j \sum_k |u_k \Sigma_{jk}| \\
&\leq 2 \|\Sigma\|_{(2,\infty)} \|\mathbf{x} - \mathbf{y}\|_2.
\end{aligned}$$

Recall that $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_n)^T \in \mathbb{R}^{n \times d}$ with iid $\mathbf{X}_i \sim N(0, \Sigma)$, so that the matrix $\mathbf{Z} = \mathbf{X} \Sigma^{-1/2}$ has iid $N(0, 1)$ entries. Since \mathbf{X}_i are iid vectors, we may write $\mathbf{M}_G = \mathbb{E} \mathbf{G}(\mathbf{X}_1)$. Let $\mathbf{Z}_i = \Sigma^{-1/2} \mathbf{X}_i$. Define a function $f: \mathbb{R}^{n \times d} \rightarrow \mathbb{R}$ by

$$f(\mathbf{Z}) = \|(\mathbb{E}_n - \mathbb{E})\mathbf{G}\|_S = \left\| \frac{1}{n} \sum_{i=1}^n \left\{ \mathbf{G}(\Sigma^{1/2} \mathbf{Z}_i) - \mathbf{M}_G \right\} \right\|_S.$$

For matrices $\mathbf{V} = (\mathbf{V}_1, \dots, \mathbf{V}_n)^T$ and $\mathbf{W} = (\mathbf{W}_1, \dots, \mathbf{W}_n)^T$ in $\mathbb{R}^{n \times d}$, we have

$$\begin{aligned}
|f(\mathbf{V}) - f(\mathbf{W})| &= \left| \left\| \frac{1}{n} \sum_{i=1}^n \mathbf{G}(\Sigma^{1/2} \mathbf{V}_i) - \mathbf{M}_G \right\|_S - \left\| \frac{1}{n} \sum_{i=1}^n \mathbf{G}(\Sigma^{1/2} \mathbf{W}_i) - \mathbf{M}_G \right\|_S \right| \\
&\leq \frac{1}{n} \sum_{i=1}^n \|\mathbf{G}(\Sigma^{1/2} \mathbf{V}_i) - \mathbf{G}(\Sigma^{1/2} \mathbf{W}_i)\|_S \\
&\leq 2 \|\Sigma\|_{(2,\infty)} \frac{1}{n} \sum_{i=1}^n \|\Sigma^{1/2} \mathbf{V}_i - \Sigma^{1/2} \mathbf{W}_i\|_2 \\
&\leq 2 \frac{\|\Sigma\|_{(2,\infty)} \|\Sigma\|_S^{1/2}}{\sqrt{n}} \|\mathbf{V} - \mathbf{W}\|_F.
\end{aligned}$$

We have here a Lipschitz continuity in nd variables. An application of the concentration inequality for Lipschitz continuous functions yields that for any $t > 0$

$$\mathbb{P} \left(f(\mathbf{Z}) - \mathbb{E} f(\mathbf{Z}) > 2 \|\Sigma\|_{(2,\infty)} \|\Sigma\|_S^{1/2} \frac{t}{\sqrt{n}} \right) \leq \exp \{-t^2/2\}$$

with $f(\mathbf{Z}) = \|(\mathbb{E}_n - \mathbb{E})\mathbf{G}\|_S = \|\Delta_n^{(1)} - \Delta_n^{(0)}\|_S$. From (3.10) it follows that

$$\mathbb{E}^2 f(\mathbf{Z}) \leq \mathbb{E} \|\Delta_n^{(1)} - \Delta_n^{(0)}\|_S^2 \leq C_1^2 \sum_{j \neq k} \frac{\Sigma_{jk}^2}{n} + \frac{4d}{45n} \leq \frac{C_1^2 \|\Sigma\|_F^2 - 2d}{n},$$

where $C_1 = 2/\pi + 1 \leq 2$ for Kendall's tau and $C_1 \leq 1 + \sqrt{8}/\pi \leq 2$ for Spearman's rho, with $C_1^2 \geq 2 + 4/45$. \blacksquare

PROOF OF LEMMA 7. By Lemmas 1 and 2, $(\mathbf{U}_n - \mathbb{E}\mathbf{U}_n)_{jk}$ are U-statistics of order m and their kernels are uniformly bounded by 1, where $m = 2$ for Kendall's tau and $m = 3$ for Spearman's rho. Let $\mathbf{D} = (D_{jk})_{d \times d}$ with $D_{jk} = (\mathbf{U}_n - \mathbb{E}\mathbf{U}_n - m\mathbf{\Delta}_n^{(1)})_{jk}$. Since $m\mathbf{\Delta}_n^{(1)}$ is the first order Hoeffding decomposition of $(\mathbf{U}_n - \mathbb{E}\mathbf{U}_n)_{jk}$, D_{jk} is second order degenerate. Thus, by Arcones and Gine (1993), $\mathbb{P}\{|D_{jk}| > Ct/n\} \leq 4e^{-t}$ for a certain numerical constant C . This gives $\mathbb{P}\{\|\mathbf{D}\|_{\max} > Ct/n\} \leq 4d^2e^{-t}$. Because $\max_{|A| \leq s} \|\mathbf{D}_{A \times A}\|_S \leq s\|\mathbf{D}\|_{\max}$, choosing $t = s(2\log 2d + t)$ completes the proof. \blacksquare

PROOF OF LEMMA 8. We prove part (ii) only as part (i) can be found in Wegkamp and Zhao (2013). Let $x = (\pi/6)\mathbb{E}\hat{\rho}_{jk}$, $y = (\pi/6)\Delta_{jk}^\rho$ and $z = (\pi/6)(\mathbb{E}\hat{\rho}_{jk} - \rho_{jk})$, so that $\hat{\Sigma}_{jk} = 2\sin(x + y)$ and $\Sigma_{jk} = 2\sin(x - z)$. By (3.20),

$$\begin{aligned} & \left| \hat{\Sigma}_{jk}^\rho - \Sigma_{jk} - \cos((\pi/6)\rho_{jk})(\pi/3)\Delta_{jk}^\rho \right| \\ &= 2 \left| \sin(x + y) - \sin(x - z) - y \cos(x - z) \right| \\ &\leq 2|z| + y^2 \\ &\leq \frac{\pi^2}{36} |\Delta_{jk}^\rho|^2 + \frac{\pi |\Sigma_{jk}|}{3(n+1)}. \end{aligned}$$

We have $\|(|\Delta_{jk}^\rho|^2)_{A \times A}\|_S \leq s\|\mathbf{\Delta}^\rho\|_{\max}^2$ and $\|(\Sigma_{jk})_{A \times A}\|_S \leq \sqrt{s}\|\mathbf{\Sigma}\|_{(2,\infty)}$. The tail probability bound for $\|\mathbf{\Delta}^\rho\|_{\max}$ follows by applying the union bound to the Hoeffding (1963) inequality. As in Wegkamp and Zhao (2013), due to $\cos((\pi/6)\rho_{jk}) = \sqrt{1 - \Sigma_{jk}^2/4}$,

$$\left\| \left(\cos((\pi/6)\rho_{jk})\Delta_{jk}^\rho \right)_{A \times A} \right\|_S \leq \sum_{m=0}^{\infty} \left| \binom{1/2}{m} \right| 4^{-m} \|\mathbf{\Delta}^\rho\|_S.$$

This completes the proof as $\sum_{m=0}^{\infty} \left| \binom{1/2}{m} \right| 4^{-m} = 2 - \sqrt{1 - 1/4}$. \blacksquare

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